Spaces of holomorphic functions on non-balanced domains

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ABSTRACT

This paper studies the coincidence of the τ_ω and τ_δ topologies on the space of holomorphic functions defined on an open subset U of a Banach space. Dineen and Mujica proved that $\tau_\omega = \tau_\delta$ when U is a balanced open subset of a separable Banach space with the bounded approximation property. Here, we study the $\tau_\omega = \tau_\delta$ problem for several types of non-balanced domains U.

Keywords:
Holomorphic function
Nachbin topology
τ_δ topology

1. Introduction

When U is an open subset of a complex Banach space E, three topologies are usually considered on the space $H(U)$ of all holomorphic functions on U: the compact open topology τ_0, the Nachbin topology τ_ω and the bornological topology τ_δ (the definitions are given below). It is known that $\tau_0 = \tau_\omega = \tau_\delta$ if E is finite dimensional, while $\tau_0 < \tau_\omega \leqslant \tau_\delta$ if E is infinite dimensional and several researchers have been interested in characterizing those spaces E such that $\tau_\omega = \tau_\delta$. The first positive result on that problem was obtained by Dineen [6] in 1972. He proved that if E is a Banach space with an unconditional Schauder basis and U is a balanced open subset of E, then $\tau_\omega = \tau_\delta$ on $H(U)$. Soon after, Cœuré [4] proved an analogous theorem for the space $E = L^1[0, 2\pi]$. Finally, in the nineties, Dineen [7, Corollary 4.18] and Mujica [10] independently obtained the most general result about the problem that we are considering:

Theorem 1 *(Dineen, Mujica). If E is a separable Banach space with the bounded approximation property and U is a balanced open subset of E, then $\tau_\omega = \tau_\delta$ on $H(U)$.*

Let us recall that U is said to be balanced if $\lambda x \in U$ for all $x \in U$ and all $\lambda \in \mathbb{C}$ with $|\lambda| \leqslant 1$. We also recall that a separable Banach space has the bounded approximation property if and only if it is isomorphic to a complemented subspace of a Banach space with a Schauder basis.

In this work, we obtain a result similar to Theorem 1 for some non-balanced domains:

Theorem 2. *Let E be a separable Banach space with the bounded approximation property and let U be a balanced open subset in E. If A is a closed bounded subset of E such that $A \subset U$ and $U \backslash A$ is connected, then $\tau_\omega = \tau_\delta$ on $H(U \backslash A)$.*

As far as we know, this theorem gives the first examples of non-balanced open domains such that $\tau_\omega = \tau_\delta$.

Let us explain the notation that appears in this work. Throughout the article, the letter E always denotes a complex Banach space and $B_E(x, r)$ represents the open ball in E with center $x \in E$ and radius $r > 0$. A seminorm p on $H(U)$ is ported by a compact subset $K \subset U$ if for every open subset V with $K \subset V \subset U$ there is a constant $C > 0$ such that

$$p(f) \leqslant C \sup_{x \in V} |f(x)|$$

for all $f \in H(U)$. The Nachbin topology τ_ω is the locally convex topology on $H(U)$ defined by all the seminorms ported by the compact subsets of U.

The symbol τ_δ represents the locally convex topology on $H(U)$ defined by all the seminorms p on $H(U)$ with the following property: for each increasing countable open cover of U, $(V_n)_{n=1}^{\infty}$, there exist $n_0 \in \mathbb{N}$ and $C > 0$ such that

$$p(f) \leqslant C \sup_{x \in V_{n_0}} |f(x)|$$

for all $f \in H(U)$. The reader is referred to the book of Mujica [11] for the main properties of holomorphic functions on infinite dimensional spaces and to book of Dineen [7] for an extensive study of the τ_ω and the τ_δ topologies.

2. Hartogs's Theorem in Banach spaces

Our study about the $\tau_\omega = \tau_\delta$ problem on $H(U \backslash A)$ strongly depends on the existence of holomorphic extension from $U \backslash A$ to U and that has led us to consider generalizations of the classical Hartogs's Theorem. This result states that if U is an open subset of $\mathbb{C}^n$ with $n \geqslant 2$, K is a compact subset of U and $U \backslash K$ is connected, then for every $f \in H(U \backslash K)$ there is $\tilde{f} \in H(U)$ such that $\tilde{f} = f$ on $U \backslash K$. In 1968, Alexander [1, p. 48] proved the same result when U is a bounded domain in a Banach space E with dimension $\dim(E) \geqslant 2$. Later, in 1985, Mujica [11, Theorem 22.6] proved it when E is a separable Hilbert space; in this last case, U can be unbounded. In 1970, Ramis [12, pp. 26–27] presented a general version of the theorem for all open subsets U and all Banach spaces E with $\dim(E) \geqslant 2$. However, Ramis's proof is very brief and it seems that it could contain some gaps. Because of that and for the sake of completeness, we prove here a particular version of Hartogs's Theorem that will be used in the study of the τ_ω and τ_δ topologies. We will need the following two results.

Theorem 3. *If $n \in \mathbb{N}$ and $M = \mathbb{R}^n$ or $M = \mathbb{C}^n$, then every balanced open subset of M is homeomorphic to M.*

Proof. The proof in the real case appears in Berger [2, Theorem 11.3.6.1]. The complex case can be deduced if $\mathbb{C}^n$ is topologically identified with $\mathbb{R}^{2n}$. $\square$

Theorem 4 *(Zorn). Let U be a connected open subset of a Banach space E. A function $f : U \to \mathbb{C}$ is holomorphic on U if both f is continuous at a point of U and $f|_{M \cap U}$ is holomorphic on $M \cap U$ for every finite dimensional subspace $M \subset E$.*

Proof. See Dineen [7, Example 3.8b]. $\square$

Theorem 5. *Let U be a balanced open subset of a Banach space E with $\dim(E) \geqslant 2$. Let A be a closed bounded subset of E such that $A \subset U$ and $U \backslash A$ is connected. If $f \in H(U \backslash A)$, then there is a unique $\tilde{f} \in H(U)$ such that $\tilde{f} = f$ on $U \backslash A$.*

Proof. Let M be any subspace of E with $2 \leqslant \dim(M) < \infty$. The set

$$V = M \cap U$$

is balanced and open in M. The set

$$A' = M \cap A$$

is closed and bounded in M, so A' is a compact subset of V. By Theorem 3, there is a homeomorphism $\varphi : V \to M$. Hence $\varphi(A')$ is a compact subset of M. If $R = \max_{x \in A'} \|\varphi(x)\|$, then $K = \varphi^{-1}(\overline{B_M}(0, R))$ is compact and

$$A' \subset K \subset V.$$

Since $M \backslash \overline{B_M}(0, R)$ is connected, $V \backslash K$ is connected as well. By Hartogs's Theorem for finite dimensional spaces, there is $g_M \in H(V)$ such that $g_M = f$ on $V \backslash K$. We define

$$\tilde{f} = g_M \quad \text{on } V.$$

The definition of $\tilde{f}$ does not depend on the choice of M. Indeed, let M and N be subspaces of E with $2 \leqslant \dim(M) < \infty$ and $2 \leqslant \dim(N) < \infty$. Let $L = M + N$. As we have seen previously, there are compact subsets K_1, K_2 and K_3 such that

$$\begin{aligned} M \cap A \subset K_1 \subset V_1 = M \cap U, \\ N \cap A \subset K_2 \subset V_2 = N \cap U, \\ L \cap A \subset K_3 \subset V_3 = L \cap U. \end{aligned}$$

There are also holomorphic functions $g_M \in H(V_1)$, $g_N \in H(V_2)$ and $g_L \in H(V_3)$ such that

$$\begin{aligned} g_M &= f \quad \text{on } V_1 \backslash K_1, \\ g_N &= f \quad \text{on } V_2 \backslash K_2, \\ g_L &= f \quad \text{on } V_3 \backslash K_3. \end{aligned}$$

The set $M \cap K_3$ is contained in V_1:

$$M \cap K_3 \subset M \cap V_3 = M \cap L \cap U = M \cap U = V_1.$$

Then $K_1 \cup (M \cap K_3)$ is a compact subset of V_1, so $V_1 \backslash (K_1 \cup (M \cap K_3)) \neq \emptyset$. Hence

$$g_L = f = g_M \quad \text{on } V_1 \backslash \big(K_1 \cup (M \cap K_3)\big) \neq \emptyset.$$

By the Identity Theorem, we have that $g_L = g_M$ on V_1. Similarly, $g_L = g_N$ on V_2. Therefore, if $x \in M \cap N \cap U = V_1 \cap V_2$, then

$$g_M(x) = g_L(x) = g_N(x).$$

This proves that the definition of $\tilde{f}$ does not depend on the choice of the finite dimensional subspace M.

By the definition of $\tilde{f}$, if M is a subspace of E such that $2 \leqslant \dim(M) < \infty$, then $\tilde{f}|_{M \cap U} = g_M \in H(M \cap U)$. By Theorem 4, in order to show that $\tilde{f}$ is holomorphic on U, we only have to prove that $\tilde{f}$ is continuous at a point of U. Let $r > 0$ be such that $A \subset \overline{B_E}(0, r)$ and let $e \in E$ with $\|e\| = 1$. The set

$$\big\{t \geqslant 0\colon\ te \in A \cup \{0\}\big\}$$

is closed and non-empty in $[0, r]$. Therefore, there exists

$$\alpha = \max\big\{t \geqslant 0\colon\ te \in A \cup \{0\}\big\}.$$

Let

$$\beta = \sup\{t > 0\colon\ te \in U\}$$

(β could be infinite). Since U is open, it follows that $\beta e \notin U$. As $\alpha e \in A \cup \{0\} \subset U$, we deduce that $\alpha < \beta$. Since U is balanced, if $\alpha < t < \beta$, then $te \in U \backslash A$, so there is $r_t > 0$ such that $B_E(te, r_t) \subset U \backslash A$. The set

$$W = \bigcup_{\alpha < t < \beta} B_E(te, r_t)$$

is open and it is contained in $U \backslash A$.

Let $x \in W$. If M is a two dimensional subspace of E such that $x, e \in M$, $V = M \cap U$ and $A' = M \cap A$, we have seen previously that there is a compact subset $K \subset V$ and there is $g_M \in H(V)$ such that $A' \subset K \subset V$ and $g_M = f$ on $V \backslash K$. Let

$$W' = \bigcup_{\alpha < t < \beta} B_M(te, r_t).$$

Then

$$W' = M \cap W \subset M \cap (U \backslash A) = V \backslash A'.$$

Let us remark that W' is connected because it is the union of the segment $\{te\colon\ \alpha < t < \beta\}$ with the balls $B_M(te, r_t)$ centered at the points of the segment.

As K is a compact subset of U and $\beta e \notin U$, there is $t_0 < \beta$ such that $t_0 e \notin K$, that is, $W' \backslash K \neq \emptyset$. The functions $f|_{V \backslash A'}$ and g_M are holomorphic on $V \backslash A'$ and $g_M = f$ on $V \backslash K$, so $g_M = f$ on $W' \backslash K$. By the Identity Theorem, $g_M = f$ on W'. Since $x \in W \cap M = W'$, it follows that

$$\tilde{f}(x) = g_M(x) = f(x).$$

Therefore, $\tilde{f}(x) = f(x)$ for every $x \in W$, which implies the continuity of $\tilde{f}$ on W. By Theorem 4, $\tilde{f}$ is holomorphic on U.

The functions $\tilde{f}$ and f are holomorphic on $U \backslash A$ and it was proved that $\tilde{f} = f$ on W. By hypothesis, $U \backslash A$ is connected, so $\tilde{f} = f$ on $U \backslash A$. The uniqueness of $\tilde{f}$ is a consequence of U being connected. $\square$

Corollary 6. *Let U be an open subset of a Banach space E with $\dim(E) \geqslant 2$. Let A be closed bounded subset of E with the following property: there is a balanced open subset V in E such that $A \subset V \subset U$ and $V \backslash A$ is connected. Then every holomorphic function on $U \backslash A$ has a unique holomorphic extension to U.*

Proof. By Theorem 5, if $f \in H(U \backslash A)$, then there is $\tilde{f} \in H(V)$ such that $\tilde{f} = f$ on $V \backslash A$. Thus, the function

$$g = \begin{cases} f & \text{on } U \backslash A, \\ \tilde{f} & \text{on } V \end{cases}$$

is well defined and is holomorphic on U. Moreover, if h is another holomorphic function on U such that $h = f$ on $U \backslash A$, then $h = f = \tilde{f}$ on $V \backslash A$. As V is connected, $h = \tilde{f}$ on V. Therefore, $h = g$ on U. $\square$

3. The τ_ω and the τ_δ topologies on $H(U)$

In order to study the coincidence of the τ_ω and the τ_δ topologies on $H(U)$, we apply several already known theorems about holomorphic extensions, the envelope of holomorphy and the spectrum of $H(U)$. Let us recall the definition of these notions. If τ is a topology on $H(U)$, the spectrum of $(H(U), \tau)$, denoted by $Spec(H(U), \tau)$, is the set of all non-zero multiplicative linear functions from $H(U)$ into $\mathbb{C}$ which are τ-continuous.

A Riemann domain over a Banach space E is a pair (X, π) where X is a Hausdorff topological space and $\pi : X \to E$ is a local homeomorphism. That means that for every $x \in X$ there is an open set Ω in X such that $x \in \Omega$, $\pi(\Omega)$ is open in E and $\pi|_\Omega : \Omega \to \pi(\Omega)$ is a homeomorphism. A function $f : X \to \mathbb{C}$ is said to be holomorphic if $f \circ (\pi|_\Omega)^{-1} \in H(\pi(\Omega))$ for every open subset $\Omega \subset X$ such that $\pi|_\Omega : \Omega \to \pi(\Omega)$ is a homeomorphism.

Let U be an open subset of E and let (X, π) be a Riemann domain over E. A continuous mapping $\varphi : U \to X$ is said to be a holomorphic extension of U if $\pi(\varphi(x)) = x$ for all $x \in U$ and for every $f \in H(U)$ there is an unique $g \in H(X)$ such that $g \circ \varphi = f$. The envelope of holomorphy of U is a Riemann domain $(\mathcal{E}(U), \pi)$ over E with the following properties:

(a) There is a holomorphic extension $\varphi : U \to \mathcal{E}(U)$.
(b) If (X_1, π_1) is another Riemann domain and $\varphi_1 : U \to X_1$ is also a holomorphic extension, then there is a continuous mapping $\psi : X_1 \to \mathcal{E}(U)$ such that the following diagrams are commutative:

$$\begin{tikzcd} U \arrow[rr, "\varphi"] \arrow[dr, "\varphi_1"'] & & \mathcal{E}(U) \\ & X_1 \arrow[ur, "\psi"'] & \end{tikzcd} \qquad \begin{tikzcd} X_1 \arrow[rr, "\psi"] \arrow[dr, "\pi_1"'] & & \mathcal{E}(U) \arrow[dl, "\pi"] \\ & E & \end{tikzcd}$$

The envelope of holomorphy of U always exists and is unique up to isomorphism (see Mujica [11, Theorem 56.4]). The following theorems relate holomorphic extensions to the spectrum of $H(U)$.

Theorem 7 *(Mujica, Schottenloher). If E is a separable Banach space with the bounded approximation property and U is a connected open subset of E, then $\mathcal{E}(U) = Spec(H(U), \tau_0)$.*

Proof. See Mujica [11, Corollary 58.10]. $\square$

Theorem 8. *Let U and $\widetilde{U}$ be connected open subsets of a Banach space such that $U \subset \widetilde{U}$. Let us assume that for every $f \in H(U)$ there is $\tilde{f} \in H(\widetilde{U})$ with $\tilde{f} = f$ on U.*

1. *The mapping*

$$T : f \in \big(H(U), \tau\big) \mapsto \tilde{f} \in \big(H(\widetilde{U}), \tau\big)$$

   is a topological isomorphism for $\tau = \tau_\delta$.
2. *If* $\widetilde{U} \subset \mathit{Spec}(H(U), \tau_\omega)$, *then the mapping* T *is also a topological isomorphism for* $\tau = \tau_\omega$.

Proof. The first statement is due to Cœuré [3] in the case of separable Banach spaces and to Hirschowitz [8] in the general case. The second statement is due to Dineen [6]. $\square$

Let us mention that Theorem 8 does not hold in general if $\widetilde{U}$ is not contained in $\mathit{Spec}(H(U), \tau_\omega)$. Josefson in [9] gave an example of a Banach space E and open subsets $U \subset \widetilde{U} \subset E$ such that the mapping T is not continuous for $\tau = \tau_\omega$.

Theorem 9. *Let* E *be a separable Banach space with the bounded approximation property. Let* U *and* $\widetilde{U}$ *be connected open subsets of* E *such that* $U \subset \widetilde{U}$ *and suppose that for every* $f \in H(U)$ *there is* $\tilde{f} \in H(\widetilde{U})$ *with* $\tilde{f} = f$ *on* U. *Then* $\tau_\omega = \tau_\delta$ *on* $H(U)$ *if and only if* $\tau_\omega = \tau_\delta$ *on* $H(\widetilde{U})$.

Proof. Let $(\mathcal{E}(U), \pi)$ be the envelope of holomorphy of U. By hypothesis, the inclusion $U \to \widetilde{U}$ is a holomorphic extension. Hence there is a continuous mapping $\psi : \widetilde{U} \to \mathcal{E}(U)$ such that the following diagrams are commutative:

$$\begin{tikzcd} U \arrow[rr, "\varphi"] \arrow[dr] & & \mathcal{E}(U) \\ & \widetilde{U} \arrow[ur, "\psi"'] & \end{tikzcd} \qquad \begin{tikzcd} \widetilde{U} \arrow[rr, "\psi"] \arrow[dr] & & \mathcal{E}(U) \arrow[dl, "\pi"] \\ & E & \end{tikzcd}$$

The mappings from U into $\widetilde{U}$ and from $\widetilde{U}$ into E are the inclusions. If $x, y \in \widetilde{U}$ and $\psi(x) = \psi(y)$, then

$$x = \pi\big(\psi(x)\big) = \pi\big(\psi(y)\big) = y.$$

This shows that ψ is injective. As ψ is continuous and its inverse $\psi^{-1} = \pi|_{\psi(\widetilde{U})}$ is also continuous, $\widetilde{U}$ can be identified with a subset of $\mathcal{E}(U)$. By Theorem 7, $\mathcal{E}(U) = \mathit{Spec}(H(U), \tau_0)$ and then

$$\widetilde{U} \subset \mathcal{E}(U) = \mathit{Spec}\big(H(U), \tau_0\big) \subset \mathit{Spec}\big(H(U), \tau_\omega\big).$$

By Theorem 8, the mapping

$$f \in \big(H(U), \tau\big) \mapsto \tilde{f} \in \big(H(\widetilde{U}), \tau\big)$$

is a topological isomorphism for $\tau = \tau_\omega$ and $\tau = \tau_\delta$. Therefore, $\tau_\omega = \tau_\delta$ on $H(U)$ if and only if $\tau_\omega = \tau_\delta$ on $H(\widetilde{U})$. $\square$

Corollary 10. *Let* E *be a separable Banach space with the bounded approximation property and let* U *be a balanced open subset of* E. *If* A *is a closed bounded subset of* E *such that* $A \subset U$ *and* $U \backslash A$ *is connected, then* $\tau_\omega = \tau_\delta$ *on* $H(U \backslash A)$.

Proof. If $\dim(E) < \infty$, then $\tau_0 = \tau_\omega = \tau_\delta$ on $H(U \backslash A)$. If $\dim(E) = \infty$, the result can be directly deduced from Theorems 1, 5 and 9. $\square$

The following theorem does not require the space E to be separable or to have the bounded approximation property.

Theorem 11. *Let A be a closed bounded subset of a Banach space E such that $E\backslash A$ is connected. Then $\tau_\omega = \tau_\delta$ on $H(E)$ if and only if $\tau_\omega = \tau_\delta$ on $H(E\backslash A)$.*

Proof. If $\dim(E) < \infty$, then $\tau_0 = \tau_\omega = \tau_\delta$ on $H(E\backslash A)$ and $\tau_0 = \tau_\omega = \tau_\delta$ on $H(E)$. Hence we can assume that $\dim(E) = \infty$. By Theorem 5, each $f \in H(E\backslash A)$ has an extension $\tilde{f} \in H(E)$. By Theorem 8, the mapping

$$T : f \in \big(H(E\backslash A), \tau\big) \mapsto \tilde{f} \in \big(H(E), \tau\big)$$

is a topological isomorphism for $\tau = \tau_\delta$. We will prove that T is also a topological isomorphism for $\tau = \tau_\omega$.

Let us suppose that p be a continuous seminorm on $(H(E), \tau_\omega)$, ported by a compact subset $K \subset E$. By hypothesis, there is $r > 0$ such that $A \subset B_E(0, r)$. Let

$$K_1 = \big\{x \in K\colon\ \|x\| \geqslant r\big\},$$
$$K_2 = \big\{x \in K\colon\ \|x\| \leqslant r\big\}.$$

Note that K_1 and K_2 are compact subsets of E, that $K = K_1 \cup K_2$ and

$$K_1 \subset E\backslash B_E(0, r) \subset E\backslash A.$$

We choose $e \in E$ such that $\|e\| = 1$ and define the following compact subset of E:

$$\widetilde{K}_2 = \big\{x + \lambda e\colon\ x \in K_2,\ \lambda \in \mathbb{C} \text{ and } |\lambda| = 2r\big\}.$$

If $x + \lambda e \in \widetilde{K}_2$, then

$$\|x + \lambda e\| \geqslant \|\lambda e\| - \|x\| \geqslant 2r - r = r.$$

This implies that

$$\widetilde{K}_2 \subset E\backslash B_E(0, r) \subset E\backslash A.$$

Therefore, $K_1 \cup \widetilde{K}_2$ is a compact subset of $E\backslash A$.

We will prove that the seminorm

$$f \in H(E\backslash A) \mapsto p \circ T(f) = p(\tilde{f})$$

is ported by $K_1 \cup \widetilde{K}_2$. Let V be an open subset of E such that

$$K_1 \cup \widetilde{K}_2 \subset V \subset E\backslash A.$$

Then there is an open neighborhood of zero $W \subset E$ such that

$$(K_1 \cup \widetilde{K}_2) + W \subset V.$$

As $K + W$ is an open neighborhood of K and p is ported by K, there exists $C > 0$ such that

$$p(g) \leqslant C \sup_{z \in K + W} \big|g(z)\big| = C \sup_{z \in (K_1 \cup K_2) + W} \big|g(z)\big| \tag{1}$$

for all $g \in H(E)$. As $K_1 + W$ is contained in V,

$$\sup_{z \in K_1 + W} \big|g(z)\big| \leqslant \sup_{z \in V} \big|g(z)\big| \tag{2}$$

for all $g \in H(E)$.

Now we are going to prove that also

$$\sup_{z \in K_2 + W} \big|g(z)\big| \leqslant \sup_{z \in V} \big|g(z)\big|$$

for all $g \in H(E)$. Let $g \in H(E)$, $x \in K_2$ and $y \in W$. The function of one complex variable

$$h(\lambda) = g(x + y + \lambda e)$$

is holomorphic on $\mathbb{C}$. By the Maximum Modulus Theorem,

$$\begin{aligned} \big|g(x+y)\big| = \big|h(0)\big| &\leqslant \sup_{|\lambda|=2r} \big|h(\lambda)\big| = \sup_{|\lambda|=2r} \big|g(x+y+\lambda e)\big| \\ &\leqslant \sup_{z \in \widetilde{K}_2 + W} \big|g(z)\big| \leqslant \sup_{z \in V} \big|g(z)\big|. \end{aligned}$$

This holds for all $x \in K_2$ and all $y \in W$, so

$$\sup_{z \in K_2 + W} \big|g(z)\big| \leqslant \sup_{z \in V} \big|g(z)\big|. \tag{3}$$

By the inequalities (1), (2) and (3),

$$p(g) \leqslant C \sup_{z \in V} \big|g(z)\big|$$

for all $g \in H(E)$.

Given $f \in H(E \backslash A)$, let $\tilde{f} \in H(E)$ such that $\tilde{f} = f$ on $E \backslash A$. Then

$$p \circ T(f) = p(\tilde{f}) \leqslant C \sup_{z \in V} \big|\tilde{f}(z)\big| = C \sup_{z \in V} \big|f(z)\big|.$$

This proves that the seminorm

$$f \in H(E \backslash A) \mapsto p \circ T(f)$$

is ported by $K_1 \cup \widetilde{K}_2$, so $p \circ T$ is continuous on $(H(E \backslash A), \tau_\omega)$. That holds for every continuous seminorm p on $(H(E), \tau_\omega)$. Therefore, the mapping T is continuous for $\tau = \tau_\omega$.

Finally, it is easy to check that the inverse

$$T^{-1} : g \in \big(H(E), \tau_\omega\big) \mapsto g|_{E \backslash A} \in \big(H(E \backslash A), \tau_\omega\big)$$

is continuous. Indeed, if q is a seminorm on $H(E \backslash A)$ ported by a compact subset $K' \subset E \backslash A$, then $q \circ T^{-1}$ is also ported by K'. Thus, T is a topological isomorphism for $\tau = \tau_\omega$ and for $\tau = \tau_\delta$. Therefore, $\tau_\omega = \tau_\delta$ on $H(E \backslash A)$ if and only if $\tau_\omega = \tau_\delta$ on $H(E)$. $\square$

Corollary 12. *If A is a closed bounded subset of ℓ_∞ such that $\ell_\infty \backslash A$ is connected, then $\tau_\omega \neq \tau_\delta$ on $H(\ell_\infty \backslash A)$.*

Proof. Dineen [5] proved that $\tau_\omega \neq \tau_\delta$ on $H(\ell_\infty)$ and then the result follows from Theorem 11. $\square$

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